# A Shock Wave Model of Unstable Rocket Combustors

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**A periodic solution describing the nonlinear waveform of the fundamental longitudinal mode of combustion instability has been obtained. The waveform consists of shock discontinuities followed by exponential decays in pressure and gas velocity. The theory predicts an explicit relationship between the functional nature of the combustion laws and the functional form of the wave shape in the combustion chamber. Therefore, this allows the possibility of investigating the forcing function or driving mechanism of the instability in a quantitative manner by experimentally determining the wave shape in the chamber. The theory applies to any instability where the physical phenomenon that provides that the forcing function contains no phasing effect. Specifically, a model of the combustion zone dynamics has been studied which considers chemical kinetics as the important factor in the forcing function. The results of the analysis on this model have been compared with the experimental results on the Princeton gas rocket research program. The method may be applied to other forms of the forcing function as well.**

## Nomenclature

$x$ = longitudinal space dimension
$t$ = time
$u$ = velocity
$a$ = speed of sound
$T$ = temperature
$p$ = pressure
$\rho$ = density
$\alpha, \beta$ = characteristic coordinates
$P, Q$ = Riemann invariants
$\gamma$ = ratio of specific heats
$\lambda^*$ = thermal conductivity
$\mathcal{E}_h$ = turbulent exchange coefficient for heat transfer
$r$ = energy release rate per unit volume
$V$ = shock velocity
$\mathrm{T}$ = period of oscillation
$\dot{m}$ = mass flow per unit area per unit time
$\epsilon$ = perturbation parameter
$R$ = inhomogeneous part of boundary condition at combustion zone
$F, G$ = homogeneous solution to equations for $x$ and $t$
$\epsilon\Delta\alpha$ = "slippage" of characteristics at shock $CD$
$\omega$ = parameter related to combustion process
$\delta$ = parameter related to combustion process
$\nu$ = $[(\gamma - 1)/2]u_0$

***Subscript***

$i$ = 0, 1, 2 indicate coefficient of $i$th term in series expansion in parameter. $AB$, $BC$, $CD$ indicate direction of shock (see Fig. 1)

## Introduction

PREVIOUSLY, the solution of the nonlinear wave equation applied to an unstable rocket combustor always has been obtained by numerical integration only. However, the combination of a new mathematical technique and certain simplifying assumptions allows the attainment of the first approximation to a periodic solution describing the waveform of the fundamental longitudinal mode. The waveform will be shown to consist of shock discontinuities followed by exponential decays in pressure and velocity. The solution explicitly relates the waveform and the wave amplitude to the means of energy release and feedback. Here, the specific case of a one-dimensional combustion zone where chemical kinetics provides the driving mechanism of the instability is investigated. The method, however, is readily extendible to other cases. This allows the possibility of investigating the driving mechanism of the instability by observing the waveform in the combustion chamber.

For analysis of nonlinear wave phenomena, it is convenient to use the characteristic coordinate perturbation technique. This involves perturbing time and space coordinates, as well as the flow properties in some amplitude parameter, with the characteristics of the field forming the new coordinate system so that $x$ and $t$ are now dependent variables. The purpose of the perturbation scheme is to obtain an infinite set of linear equations from a finite set of nonlinear equations. The characteristic coordinate perturbation scheme yields a greater portion of homogeneous equations than an ordinary perturbation scheme does. In addition, it allows an operation with functions that may be continued through shock discontinuities. Riemann invariants are an example of these functions. This abstract technique may not be as straightforward conceptually as the normal perturbation technique, but it does save much analytical work.

The characteristic coordinate perturbation technique was developed by Lighthill,[1] Lin,[2] and Fox,[3] and was first applied to the problem of cyclic oscillations with shock waves by Chu.[4,5] However, his oscillations occurred in a one-dimensional chamber where energy but no mass was added at one end, and a solid wall stood at the other end so that the mean flow was zero. The problem becomes considerably more complicated for an analysis of a rocket chamber. There is both energy and mass addition due to combustion which, of course, is accompanied by a mean flow. Also, the nozzle boundary condition involves not only wave reflection as the solid wall boundary does, but also the dispersion of the wave and the convection of a portion of its energy out of the chamber.

Note that this paper introduces simplifications to the technique which involve the continuation of solutions across shock waves. These simplifications allow a clearer insight to the physical nature of the phenomena, thereby permitting the attack of more complex problems.

## Chamber Gasdynamics

A shock-wave model of the fundamental mode of combustion instability is investigated. The assumption is made

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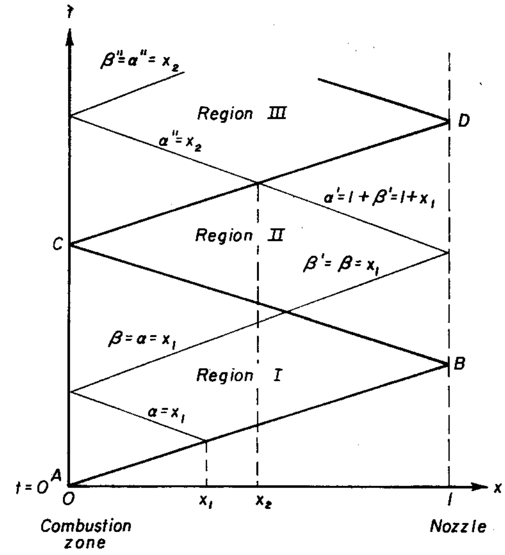


**Fig. 1 Coordinate numbering system.**

that the characteristic time of the combustion process is negligible as compared to the wave travel time. Other assumptions are as follows:

1) The flow is one-dimensional.
2) The chamber cross-sectional area is constant.
3) The chamber is very long so that the configuration may be approximated by the limiting case of zero-length nozzle and concentrated combustion zone at the chamber end.
4) A shock wave moves back and forth the length of the chamber with a constant period, reflecting alternately from the nozzle end and the combustion end.
5) Flow is homentropic outside of the combustion zone up to and including second order in the wave amplitude. This allows shock waves to occur, but no entropy waves are allowed.
6) The chamber gas is calorically perfect.

If the length of the combustion zone is small as compared to the chamber length (as assumed previously), unstable operation may be considered forced in a piston-like manner by the combustion process. The power per unit area (pressure times gas velocity) at the end of the combustion zone is the rate at which work is done upon the burned chamber gases. This power term is related to the rate of energy release within the combustion zone and therefore, is, in general, a function of thermodynamic conditions within the zone. If appropriate space-wise mean values are used for the thermodynamic properties (only within the combustion zone), and further, if all thermodynamic properties are related to the speed of sound, a relation‡ applicable at the end of the combustion zone is obtained

$$\frac{u - u_0}{u_0} = \omega\left(\frac{a - a_0}{a_0}\right) + \delta\left(\frac{a - a_0}{a_0}\right)^2 + \text{terms of order}\left(\frac{a - a_0}{a_0}\right)^3 \quad (1)$$

The constants $\omega$ and $\delta$ may be calculated based on a knowledge of the combustion process. Later, they will be determined for a one-dimensional combustion zone where perturbations due to chemical effects are important, but perturbations due to diffusion effects are negligible. The preceding relation is the boundary condition at one chamber end when the limiting case of concentrated combustion is considered.

All thermodynamic variables are nondimensionalized with respect to their steady-state values. The gas velocity is nondimensionalized with respect to the steady-state speed of sound, space dimension with respect to chamber length, and time dimension with respect to chamber length divided by speed of sound.

‡ Zero subscript denotes a steady-state value.

The well-known compatibility relations may be obtained from the equations of unsteady, one-dimensional motion for a fluid. Under our assumptions, these relations (to second order) are

$$[2/(\gamma - 1)]da + du = 0 \text{ along } dx/dt = u + a$$

$$[2/(\gamma - 1)]da - du = 0 \text{ along } dx/dt = u - a$$

Integration of $dx/dt$ gives the two families of characteristic lines. Now, let each member of the family with negative slope $dx/dt = u - a$ be characterized by a certain value of the parameter $\alpha$ and each member of the family with positive slope $dx/dt = u + a$ be characterized by a certain value of the parameter $\beta$. Let $\alpha$ and $\beta$ become the new independent variables, and it is said

$$u = u(\alpha, \beta) \qquad a = a(\alpha, \beta)$$

$$x = x(\alpha, \beta) \qquad t = t(\alpha, \beta)$$

The compatibility relations are now partial differential equations in $\alpha$, $\beta$ coordinates

$$\begin{aligned} [2/(\gamma - 1)](\partial a/\partial \alpha) + (\partial u/\partial \alpha) &= 0 \\ [2/(\gamma - 1)](\partial a/\partial \beta) - (\partial u/\partial \beta) &= 0 \\ \partial x/\partial \alpha &= (u + a)(\partial t/\partial \alpha) \\ \partial x/\partial \beta &= (u - a)(\partial t/\partial \beta) \end{aligned} \quad (2)$$

It is necessary to have a numbering system for the $\alpha$, $\beta$ coordinates. A convenient numbering system is shown in Fig. 1. Region I is the interior of the triangle $ABC$ while region II is the interior of the triangle $BCD$. Primes are used for region II and double primes are used for region III. The point $t = 0$ is chosen as the time of the shock reflection at the combustion end. Number $\alpha$ by the value of $x$ at its intersection with the shock wave $AB$. Number $\beta$ by the values of $\alpha$ at their intersection at the $x = 0$ position (injector). Let $\beta' = \beta$ at their intersection at the shock $BC$. Let $\alpha' = 1 + \beta'$ at their intersection at the $x = 1$ position (nozzle). Number $\alpha''$ by the value of $x$ at its intersection with the shock $CD$, and let $\beta'' = \alpha''$ at their intersection at $x = 0$. The cyclic condition will be implied by stating that conditions along the shock $CD$ are identical to conditions along the shock $AB$. Therefore, the characteristics in the other regions need not be numbered since the solutions will be cyclic.

The dependent variables are perturbed in a regular series in $\epsilon$, which is some amplitude parameter as yet not specifically defined

$$\begin{aligned} u &= u_0 + \epsilon u_1(\alpha, \beta) + \epsilon^2 u_2(\alpha, \beta) + \ldots \\ a &= 1 + \epsilon a_1(\alpha, \beta) + \epsilon^2 a_2(\alpha, \beta) + \ldots \\ p &= 1 + \epsilon p_1(\alpha, \beta) + \epsilon^2 p_2(\alpha, \beta) + \ldots \\ x &= x_0(\alpha, \beta) + \epsilon x_1(\alpha, \beta) + \epsilon^2 x_2(\alpha, \beta) + \ldots \\ t &= t_0(\alpha, \beta) + \epsilon t_1(\alpha, \beta) + \epsilon^2 t_2(\alpha, \beta) + \ldots \end{aligned}$$

These series are substituted in the system of equations (2) and the equations are separated according to powers in $\epsilon$ as is the standard procedure. Since the flow field is uniform to lowest order, the equations up to second order become $u_0$ = const, $a_0$ = const = 1, and

$$\frac{\partial x_0}{\partial \alpha} = (u_0 + 1)\frac{\partial t_0}{\partial \alpha} \qquad \frac{\partial x_0}{\partial \beta} = (u_0 - 1)\frac{\partial t_0}{\partial \beta} \quad (3)$$

$$\frac{2}{\gamma - 1}\frac{\partial a_1}{\partial \alpha} + \frac{\partial u_1}{\partial \alpha} = 0 \qquad \frac{2}{\gamma - 1}\frac{\partial a_1}{\partial \beta} - \frac{\partial u_1}{\partial \beta} = 0 \quad (4)$$

$$\frac{\partial x_1}{\partial \alpha} = (u_0 + 1)\frac{\partial t_1}{\partial \alpha} + (u_1 + a_1)\frac{\partial t_0}{\partial \alpha}$$
$$\frac{\partial x_1}{\partial \beta} = (u_0 - 1)\frac{\partial t_1}{\partial \beta} + (u_1 - a_1)\frac{\partial t_0}{\partial \beta} \quad (5)$$

$$\frac{2}{\gamma - 1}\frac{\partial a_2}{\partial \alpha} + \frac{\partial u_2}{\partial \alpha} = 0 \qquad \frac{2}{\gamma - 1}\frac{\partial a_2}{\partial \beta} - \frac{\partial u_2}{\partial \beta} = 0 \quad (6)$$

First, Eqs. (4) and (6) for $u$ and $a$ are solved. The first- and second-order equations are similar, and so they will be solved in identical fashion. Letting the subscript $i = 1$ or 2, the solutions are

$$[2/(\gamma - 1)]a_i + u_i = P_i(\beta)$$
$$[2/(\gamma - 1)]a_i - u_i = Q_i(\alpha)$$

so that

$$u_i = [P_i(\beta) - Q_i(\alpha)]/2$$
$$a_i = [(\gamma - 1)/4][P_i(\beta) + Q_i(\alpha)] \quad (7)$$

Note that $P/2$ and $Q/2$ are the Riemann invariants.

$P_i(\beta)$ and $Q_i(\alpha)$ are still arbitrary and may be determined by knowledge of initial conditions and boundary conditions. The boundary conditions are well-defined, as will later be shown, but there is no knowledge of initial conditions. The solution will be obtained by leaving the initial conditions arbitrary and applying the cyclic conditions in their place. Once $P_i$ and $Q_i$ are determined, a simple calculation yields $a_i$ and $u_i$. Also, once these are determined, $x$ and $t$ are found from their governing differential equations and boundary conditions, which allows the transformation to the original coordinate system.

Since the nozzle is short, any oscillatory processes within it may be considered quasi-steady. Therefore, the entrance Mach number is constant since it is a function of area ratio only. The nozzle boundary condition becomes

$$u'_i = u_0 a'_i \text{ at } \alpha' = 1 + \beta'$$

or

$$Q_i' = \frac{1 - \nu}{1 + \nu} P_i' \text{ at } \alpha' = 1 + \beta' \quad (8)$$

where

$$\nu = [(\gamma - 1)/2]u_0$$

Equation (1) in nondimensional form gives the boundary condition at the combustion zone:

$$u = u_0 + \omega u_0(a - 1) + \delta u_0(a - 1)^2 + O(a - 1)^3 \quad (9)$$

This results in the following boundary condition for the first-order terms in the perturbation series $u_1 = \omega u_0 a_1$. When $\omega > 1$, the admittance at the concentrated combustion zone is larger than the nozzle admittance. This is an unstable situation as can be shown by a small perturbation analysis that indicates an exponential growth of wave amplitude when $\omega > 1$. This small perturbation analysis is standard and is performed in Appendix A. It follows from a comparison with the small perturbation analysis that an oscillation cannot be periodic to first order in $\epsilon$ if no shock is present, since the energy added to the oscillation by the interaction with the combustion process is greater than the energy taken from the oscillation by the nozzle outflow. If a periodic solution is to exist, there must be another mechanism, besides the nozzle, which removes energy of oscillation. Since a shock wave seems to be the most realistic choice of such a dissipative mechanism, its existence has been postulated. The dissipation of energy by a shock is monotonically increasing with its amplitude or strength. More specifically, the dissipation or creation of entropy is of third order in shock strength. It is then expected that the strength of the shock is monotonically increasing with the difference in the admittances of the combustion zone and the nozzle. Note, of course, that $\epsilon$ is representative of the shock strength and $(\omega - 1)$ is representative of the difference in the admittances at the chamber ends. Since series convergence cannot be proven for any choice of $\epsilon$, there is freedom in the specification of the relationship between $\epsilon$ and $(\omega - 1)$. Therefore, the most simple functional relationship will be chosen which says that $\epsilon$ is directly proportional to $(\omega - 1)$, or specifically

$$\epsilon = \nu(\omega - 1) \quad (10)$$

The boundary condition (9) at the combustion zone now becomes

$$u_i = u_0 a_i + R_i \quad \text{at} \quad x = 0$$

where

$$R_1 = 0$$
$$R_2 = [2/(\gamma - 1)]a_1 + u_0 \delta a_1^2$$

This may be put in the more convenient form

$$P_i = [(1 + \nu)/(1 - \nu)]Q_i + [2/(1 - \nu)]R_i \quad (11)$$

To first order, the boundary conditions at the chamber ends are identical so that a periodic solution may now be obtained.

As is well known[6] one family of the Riemann invariants is always continuous up to second order through a shock wave; $P$ invariants may be continued through rearward-moving shocks, whereas $Q$ invariants may be continued through forward-moving shocks. The two boundary conditions (8) and (11) and this property of continuation through shocks allow the determination of $P_i(\beta)$, $P_i'(\beta')$, and $Q_i'(\alpha')$ in terms of $Q_i(\alpha)$. It is found that

$$\left.\begin{aligned} P_i(\beta) &= \frac{1 + \nu}{1 - \nu} Q_i(\beta) + \frac{2}{1 - \nu} R_i(\beta) \\ P_i'(\beta') &= P_i(\beta) \text{ where } \beta = \beta' \\ Q_i'(\alpha') &= \frac{1 - \nu}{1 + \nu} P_i'(\alpha' - 1) = Q_i(\alpha' - 1) + \\ &\qquad \frac{2}{1 + \nu} R_i(\alpha' - 1) \end{aligned}\right\} \quad (12)$$

Note that $Q_i(\alpha)$ will be determined upon application of the cyclic condition.

When the cyclic condition is applied to determine $Q_1$, it is necessary to relate $\alpha$ and $\beta$ to $x$ and $t$. Hence, Eqs. (3) and (5) for $x$ and $t$ must be solved. The zero-order equations (3) may be put in the convenient form

$$\frac{\partial}{\partial \beta}\left[\frac{x_0}{1 - u_0} + t_0\right] = 0 \qquad \frac{\partial}{\partial \alpha}\left[\frac{x_0}{1 + u_0} - t_0\right] = 0$$

with the general solution

$$\frac{x_0}{1 - u_0} + t_0 = G_0(\alpha) \qquad \frac{x_0}{1 + u_0} - t_0 = F_0(\beta)$$

The boundary conditions state

$$x_0 = 0 \text{ at } \alpha = \beta \qquad x_0 = 1 \text{ at } \alpha' = 1 + \beta'$$
$$t_0 = 0 \text{ at } \alpha = \beta = 0 \qquad x_0 = \alpha \text{ at } \beta = 0$$

Upon their application, it is found that for region I

$$G_0(\alpha) = \frac{2\alpha}{1 - u_0^2} \qquad F_0(\beta) = -\frac{2\beta}{1 - u_0^2}$$
$$x_0 = \alpha - \beta \qquad t_0 = \frac{\alpha}{1 + u_0} + \frac{\beta}{1 - u_0} \quad (13a)$$

Using primes on $x$, $t$, $F$, $G$, $\alpha$, and $\beta$ in the above would give the solution for region II

$$G_0'(\alpha') = \frac{2\alpha'}{1 - u_0^2} \qquad F_0'(\beta') = -\frac{2\beta'}{1 - u_0^2}$$
$$x_0' = \alpha' - \beta' \qquad t_0' = \frac{\alpha'}{1 + u_0} + \frac{\beta'}{1 - u_0} \quad (13b)$$

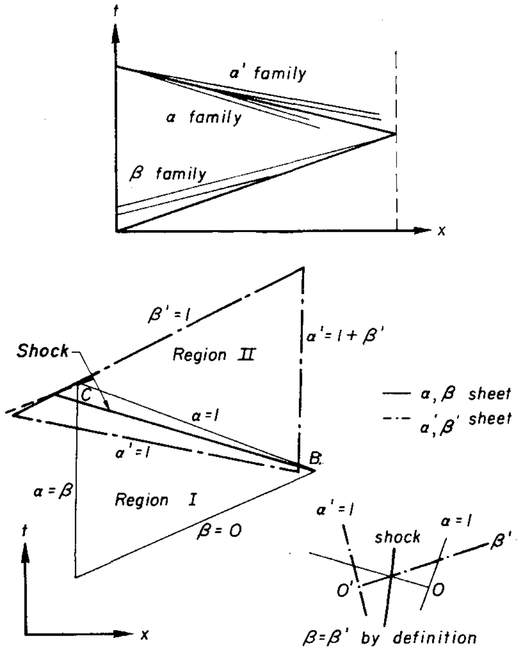


**Fig. 2 Continuation of solutions across shock waves.**

Similarly, the first-order equations (5) may be written as

$$\frac{\partial}{\partial\beta}\left[\frac{x_1}{1-u_0}+t_1\right]=\frac{3-\gamma}{4}\frac{P_1(\beta)}{(1-u_0)^2}-\frac{\gamma+1}{4}\frac{Q_1(\alpha)}{(1-u_0)^2}$$

$$\frac{\partial}{\partial\alpha}\left[\frac{x_1}{1+u_0}-t_1\right]=\frac{\gamma+1}{4}\frac{P_1(\beta)}{(1+u_0)^2}-\frac{3-\gamma}{4}\frac{Q_1(\alpha)}{(1+u_0)^2}$$

with the general solution for region I

$$\frac{x_1}{1-u_0}+t_1=G_1(\alpha)+\frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2\times\int_0^\beta P_1(z)dz-\frac{\gamma+1}{4}\frac{\beta Q_1(\alpha)}{(1-u_0)^2}$$

$$\frac{x_1}{1+u_0}-t_1=F_1(\beta)+\frac{\gamma+1}{4}\frac{\alpha P_1(\beta)}{(1+u_0)^2}-\frac{3-\gamma}{4}\frac{1}{(1+u_0)^2}\int_0^\alpha Q_1(z)dz \tag{14a}$$

where $z$ is a dummy variable. For region II, the solution is

$$\frac{x_1'}{1-u_0}+t_1'=G_1'(\alpha')+\frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2\times\int_0^{\beta'} P_1'(z)dz-\frac{\gamma+1}{4}\frac{\beta' Q_1'(\alpha')}{(1-u_0)^2}$$

$$\frac{x_1'}{1+u_0}-t_1'=F_1'(\beta')+\frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2\times\alpha'P_1'(\beta')-\frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2\int_1^{\alpha'} Q_1'(z)dz \tag{14b}$$

The initial condition would be

$$[x_1/(1-u_0)]+t_1=G_1(\alpha)\text{ at }\beta=0$$

However, the initial condition will be left arbitrary and will be replaced by a cyclic condition. That is, $G_1$ is left arbitrary. The boundary conditions are

$$\begin{aligned} t_1 &= 0 \text{ at } \alpha=\beta=0\\ x_1 &= 0 \text{ at } \alpha=\beta\\ x_1 &= 0 \text{ at } \alpha'=1+\beta' \end{aligned} \tag{15a}$$

The convenience of grouping the variables $x_1$ and $t_1$ into the two families shown previously will be demonstrated when the solutions are continued across a shock discontinuity. As is shown in Fig. 2a, characteristic lines are constantly intersecting shock waves from both sides. Of course, they no longer exist in a real fashion after their intersection. When using characteristic coordinate, it is convenient to extend the characteristics beyond their points of intersection with the shock by considering the transformation of $x$, $t$ to $\alpha$, $\beta$ as being double-valued for some small region ($O[\epsilon]$) near the shock wave. Of course, only one value of the transformation is applicable in reality, but this abstraction saves much labor. Figure 2b shows the slight overlapping of the $\alpha$, $\beta$ sheet (region I) and the $\alpha'$, $\beta'$ sheet (region II) near the rearward-moving shock $BC$. Now, the Riemann invariant $P(\beta)/2$ may be continued across the shock by saying $P_1(\beta)$ at point 0 (see Fig. 2c) equals $P_1'(\beta')$ at point 0′ where $\beta=\beta'$ by definition, since they meet at the shock wave. It is readily shown by Taylor series expansion about 0 and 0′ and a matching process at the shock that

$$\left[\frac{x_1}{1+u_0}-t_1\right]_0=\left[\frac{x_1'}{1+u_0}-t_1'\right]_{0'}+O(\epsilon) \tag{15b}$$

As shown schematically in Fig. 2b, the path of the shock $BC$ is given by the relationship

$$\alpha=1+\epsilon\eta(\beta)+O(\epsilon^2)$$

This applies in region I. The equivalent statement written for region II says

$$\alpha'=1+\epsilon\xi(\beta')+O(\epsilon^2)$$

Obviously, the function $[x/(1+u_0)-t]$ is continuous across the shock wave since $x$ and $t$ are continuous across the shock wave. It follows that this function has the same value at the shock in both region I and region II. This function may be expressed as a Taylor series in each region, and then the two series are set equal at the shock wave. In region I, the function is expanded about the line $\alpha=1$ (point 0) and in region II, about the line $\alpha'=1$ (point 0′). The final result is

$$\left[\frac{x_0}{1+u_0}-t_0\right]_0+\epsilon\left[\frac{\partial}{\partial\alpha}\left(\frac{x_0}{1+u_0}-t_0\right)\right]_0\eta(\beta)+\epsilon\left[\frac{x_1}{1+u_0}-t_1\right]_0=\left[\frac{x_0'}{1+u_0}-t_0'\right]_{0'}+\epsilon\left[\frac{\partial}{\partial\alpha'}\left(\frac{x_0'}{1+u_0}-t_0'\right)\right]_{0'}\xi(\beta')+\epsilon\left[\frac{x_1'}{1+u_0}-t_1'\right]_{0'}+O(\epsilon^2)$$

Note that $(\partial/\partial\alpha)\{[x_0/(1+u_0)]-t_0\}$ is zero everywhere, and that, as can be shown from Eq. (13),

$$\left[\frac{x_0}{(1+u_0)}-t_0\right]_0=\left[\frac{x_0'}{(1+u_0)}-t_0'\right]_{0'}$$

From this, it is seen that Eq. (15) immediately follows. Combination of Eqs. (14) and (15) reveals that

$$F_1(\beta)=F_1'(\beta')+\left(\frac{3-\gamma}{4}\right)\frac{1}{(1+u_0)^2}\int_0^1 Q_1(z)dz+O(\epsilon)$$

The advantage of this is that it is not necessary to calculate explicitly the exact location of our shock before proceeding with the determination of higher order coefficients in region II. This method has uncoupled that calculation from the rest, and it may be performed later, if so desired, providing a tremendous simplification over previous methods. Similarly, the other family or grouping of $x_1$ and $t_1$ may be continued across a forward-moving shock wave. This continuation process is not valid for $x_2$ and $t_2$ where discontinuities will appear.

The conditions (15a) and (15b) lead to the following;

$$\left.\begin{aligned}
F_1(\beta) &= -G_1(\beta) + \frac{3-\gamma}{4}\left\{\left(\frac{1}{1+u_0}\right)^2 - \frac{1+\nu}{1-\nu}\left(\frac{1}{1-u_0}\right)^2\right\}\int_0^\beta Q_1(z)dz + \\
&\quad \frac{\gamma+1}{4}\left\{\left(\frac{1}{1-u_0}\right)^2 - \frac{1+\nu}{1-\nu}\left(\frac{1}{1+u_0}\right)^2\right\}\beta Q_1(\beta) \\
F_1'(\beta) &= F_1(\beta) - \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \times \int_0^1 Q_1(z)dz + O(\epsilon) \\
G_1'(\alpha') &= G_1(\alpha'-1) + \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \times \\
&\quad \int_0^1 Q_1(z)dz - \frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2\frac{1+\nu}{1-\nu}(Q_1\alpha' - 1) + O(\epsilon)
\end{aligned}\right\} \quad (16)$$

The cyclic condition is applied by stating that flow conditions along the shock $AB$ in region I are identical to flow conditions along the shock $CD$ in region III. When the cyclic condition is applied, it is necessary to know the difference in the values of $\alpha'$ and $\alpha''$ at any point of their intersection with the shock wave $CD$.

In region III, the same differential equations (3) and (5) govern $x$ and $t$ with the boundary conditions,

$$\alpha'' = x \text{ at } \beta'' = 0 \qquad x = 0 \text{ at } \alpha'' = \beta''$$
$$t = \mathbf{T} \text{ at } \alpha'' = \beta'' = 0$$

where $\mathbf{T}$ is the period of the oscillating and is still unknown. These must result in the following for the cyclic condition to hold:

$$\begin{aligned} G_0''(\alpha'') &= G_0(\alpha'') + \mathbf{T}_0 \\ G_1''(\alpha'') &= G_1(\alpha'') + \mathbf{T}_1 \end{aligned} \quad (17)$$

where

$$\mathbf{T} = \mathbf{T}_0 + \epsilon\mathbf{T}_1 + \ldots$$

Before proceeding, the period must be determined in order to obtain the solution for region III. It is known that the period is directly related to shock velocities by the relations

$$\int_0^1 dx = \int_0^{\mathbf{T}'} V_{AB}\, dt \qquad \int_1^0 dx = \int_{\mathbf{T}'}^{\mathbf{T}'+\mathbf{T}''} V_{BC}\, dt \quad (18)$$

where $\mathbf{T} = \mathbf{T}' + \mathbf{T}''$ and $\mathbf{T}'$ is the time of forward-shock travel and $\mathbf{T}''$ is the time of rearward-shock travel. It is also known that the shock velocities may be related to the flow properties on both sides of the shock by means of the conservation equations. A well-known result is that the instantaneous velocity for a forward-traveling shock is approximately (to first order) the average of the slopes of the two $P$-characteristics (one on each side in an $x$ vs $t$ plot) intersecting the shock at that instant. For rearward-moving shocks, a similar approximation uses the slopes of $Q$ characteristics. The slopes $dx/dt = u \pm a$ may be related to $Q$ through Eqs. (7) and (12). In a convenient form,§ these approximations are,

$$\begin{aligned}
V_{AB} &= 1 + u_0 + \epsilon\left\{-\frac{3-\gamma}{4}Q_1(\alpha) + \frac{\gamma+1}{8}\frac{1+\nu}{1-\nu}[Q_1(1) + Q_1(0)]\right\} + O(\epsilon^2) \\
V_{BC} &= -1 + u_0 + \epsilon\left\{\frac{3-\gamma}{4}\frac{1+\nu}{1-\nu}Q_1(\beta) - \frac{\gamma+1}{8}[Q_1(1) + Q_1(0)]\right\} + O(\epsilon^2)
\end{aligned} \quad (19)$$

§ Note that $V_{AB}$ has been determined by using flow conditions in region I at the shock $AB$ and in region II at the shock $CD$. This is valid because of the cyclic nature of the flow.

Combination of Eqs. (18) and (19), and separation according to powers of $\epsilon$, yield the following results:

$$1 = (1+u_0)\mathbf{T}_0' \qquad -1 = (u_0 - 1)\mathbf{T}_0''$$

$$0 = (1+u_0)\mathbf{T}_1' + \int_0^1\left\{-\frac{3-\gamma}{4}Q_1(\alpha) + \frac{\gamma+1}{8}\frac{1+\nu}{1-\nu}[Q_1(1) + Q_1(0)]\right\}\frac{d\alpha}{1+u_0}$$

$$0 = (u_0 - 1)\mathbf{T}_1'' + \int_0^1\left\{\frac{3-\gamma}{4}\frac{1+\nu}{1-\nu}Q_1(\beta) - \frac{\gamma+1}{8}[Q_1(1) + Q_1(0)]\right\}\frac{d\beta}{1-u_0}$$

These four relationships determine $\mathbf{T}_0'$, $\mathbf{T}_0''$, $\mathbf{T}_1'$, and $\mathbf{T}_1''$, which, upon addition, yield the results for $\mathbf{T}_0$ and $\mathbf{T}_1$:

$$\mathbf{T}_0 = \frac{2}{1-u_0^2}$$

$$\mathbf{T}_1 = \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1+\nu}{1-\nu}\right)\left(\frac{1}{1-u_0}\right)^2\right] \times \int_0^1 Q_1(z)dz - \frac{\gamma+1}{8}\left[\left(\frac{1}{1-u_0}\right)^2 + \left(\frac{1+\nu}{1-\nu}\right)\left(\frac{1}{1+u_0}\right)^2\right][Q_1(1) + Q_1(0)]$$

These relationships for $\mathbf{T}_0$ and $\mathbf{T}_1$ are substituted into Eq. (17) with the following result:

$$\begin{aligned}
G_0''(\alpha'') &= G_0(\alpha'') + \frac{2}{1-u_0^2} \\
G_1''(\alpha'') &= G_1(\alpha'') + \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 + \frac{1+\nu}{1-\nu}\left(\frac{1}{1-u_0}\right)^2\right]\int_0^1 Q_1(z)dz - \frac{\gamma+1}{8} \times \\
&\quad \left[\left(\frac{1}{1+u_0}\right)^2\frac{1+\nu}{1-\nu} + \left(\frac{1}{1-u_0}\right)^2\right][Q_1(1) + Q_1(0)]
\end{aligned} \quad (20)$$

Now, the function $[x/(1 - u_0) + t]$ is continuous (up to and including first order) through the shock $CD$. It is evaluated in region II from Eqs. (13b, 14b, and 16), and in region III from Eqs. (13a, 14a, and 20). Note that the function $[x''/(1 - u_0) + t'']$ in region III is identical to $[x/(1 + u_0) - t]$ in region I plus a constant equal to the period of oscillation. The result of matching this function across the shock $CD$ is

$$\begin{aligned}
&\frac{2(1+\alpha'')}{1-u_0^2} + \epsilon\left\{G_1(\alpha'') + \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1+\nu}{1-\nu}\right)\left(\frac{1}{1-u_0}\right)^2\right]\int_0^1 Q_1(z)dz - \frac{\gamma+1}{8}\left[\left(\frac{1+\nu}{1-\nu}\right) \times \left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right][Q_1(1) + Q_1(0)]\right\} = \frac{2\alpha'}{1-u_0^2} + \\
&\epsilon\left\{G_1(\alpha'-1) + \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 + \frac{1+\nu}{1-\nu} \times \left(\frac{1}{1-u_0}\right)^2\right]\int_0^1 Q_1(z)dz - \frac{\gamma+1}{4}\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]Q_1(\alpha'-1)\right\} + O(\epsilon^2)
\end{aligned}$$

It is readily seen that $\alpha'' - \alpha' + 1$ is of order $\epsilon$. Therefore, $\epsilon[G_1(\alpha'') - G_1(\alpha' - 1)]$ and $\epsilon[Q_1(\alpha'') - Q_1(\alpha' - 1)]$ are both of order $\epsilon^2$. Using this, a simple relationship results for the difference in the values of $\alpha''$ and $\alpha'$ at the shock $CD$. The knowledge of this difference is necessary for the application of the cyclic condition as will later be shown. This dif-

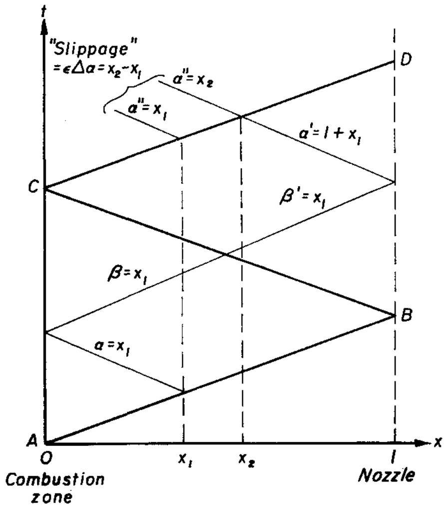


**Fig. 3 "Slippage" of characteristics.**

ference is the "slippage" of the characteristics at the shock $CD$ and is found to be the following:

$$\epsilon\Delta\alpha \equiv \alpha'' - \alpha' + 1 = \epsilon\,\frac{\gamma+1}{8}\left[\frac{1-u_0}{1+u_0}\frac{1+\nu}{1-\nu} + \frac{1+u_0}{1-u_0}\right]\left\{\frac{Q_1(1)+Q_1(0)}{2} - Q_1(\alpha'')\right\} + O(\epsilon^2)$$

This "slippage" $\epsilon\Delta\alpha$ is shown schematically in Fig. 3.

The continuation of the Riemann invariant across the shock $CD$ results in

$$[\epsilon Q_1''(\alpha'') + \epsilon^2 Q_2''(\alpha'')]_{\text{III}} = [\epsilon Q_1'(\alpha') + \epsilon^2 Q_2'(\alpha')]_{\text{II}} + O(\epsilon^3)$$

and, noting that $\alpha'' = \alpha' - 1 + \epsilon\Delta\alpha$, we get

$$[\epsilon Q_1''(\alpha'-1) + \epsilon^2\,(dQ_1''/d\alpha')\,(\alpha'-1)\Delta\alpha + \epsilon^2 Q_2''(\alpha'-1)]_{\text{III}} = [\epsilon Q_1'(\alpha') + \epsilon^2 Q_2'(\alpha')]_{\text{II}} + O(\epsilon^3)$$

The cyclic condition says that the flow in region I is identical to that in region III, so that

$$\epsilon Q_1(\alpha'-1) + \epsilon^2\,\frac{dQ_1}{d\alpha}\,(\alpha'-1)\Delta\alpha + \epsilon^2 Q_2(\alpha'-1) = \epsilon Q_1'(\alpha') + \epsilon^2 Q_2'(\alpha') + O(\epsilon^3)$$

Noting the relation between $Q_i'(\alpha')$ and $Q_i(\alpha)$ as already determined, it is found that our matching condition is trivial to first order and gives a governing ordinary differential equation for $Q_1(\alpha)$ to second order

$$(dQ_1/d\alpha)\Delta\alpha = [2/(1+\nu)]R_2(\alpha) \tag{21}$$

The solution to this equation will yield the solution of our problem. What has been done is very common in nonlinear near-resonant solutions,[7] that is, a second-order investigation has yielded a first-order solution. If rotational effects did not make it impossible, a third-order investigation would yield the second-order solution, and so forth. However, a first-order solution yields a good deal of qualitative information about the oscillation and also, for not-too-large amplitudes, the analysis is quite accurate in a quantitative manner, assuming that there exist no other errors than those related to the degree of the approximation.

If the following definitions are made,

$$Z = \frac{16}{\gamma+1}\,\frac{1}{(1-\nu^2)\left[\frac{1-u_0}{1+u_0}\frac{1+\nu}{1-\nu} + \frac{1+u_0}{1-u_0}\right]}$$

$$\lambda = \frac{16\,u_0\delta}{(\gamma+1)(1+\nu)} \times \frac{\{[(\gamma-1)/2][1/(1-\nu)]\}^2}{\{[(1-u_0)/(1+u_0)][(1+\nu)/(1-\nu)] + [(1+u_0)/(1-u_0)]\}}$$

$$c = [Q_1(1) + Q_1(0)]/2$$

then Eq. (21) becomes

$$dQ_1/d\alpha = -(ZQ_1 + \lambda Q_1^2)/(Q_1 - c) \tag{22}$$

This first-order nonlinear ordinary differential equation normally requires one boundary condition for the complete solution, but there is none. Also, the values of the function at the endpoints of the domain of interest $0 \leq \alpha \leq 1$ appear as constants in the equation. These constants are not known a priori. These two difficulties will be overcome in the following manner. It can be shown that physically reasonable solutions exist only for one value of the parameter $c$. That only $c = 0$ can give an oscillatory solution is the important result of a topological investigation of the ordinary differential equation (Appendix B). Also, stating $c = 0$ gives a relation between the values of the solution at two points $\alpha = 0$ and $\alpha = 1$. This is just as satisfactory as stating the definite value of the solution at one point, which is the usual statement of the initial or boundary condition. Hence, the following solution is obtained for Eq. (21):

$$Q_1(\alpha) = \frac{Z}{\lambda}\left[\frac{2}{1+e^{-\lambda}}\,e^{-\lambda\alpha} - 1\right] \tag{23}$$

Note that the exponential form of the solution indicates an exponential wave form. Also, as $\lambda \to 0$ ($\delta \to 0$) a sawtooth waveform is obtained as shown by the limiting case

$$Q_1(\alpha) = Z(\tfrac{1}{2} - \alpha)$$

Note that once $Q_1(\alpha)$ is determined, $P_1(\beta)$, $Q_1'(\alpha')$, and $P_1'(\beta')$ may be determined from Eq. (12) so that sufficient information is available to calculate the flow field properties from Eqs. (7) and (13) as a function of position and time. Note that the pressure perturbation may easily be related to the speed of sound perturbation by the isentropic relation. For region I, the final solutions are obtained

$$u - u_0 = \epsilon\,\frac{Z}{\lambda}\left\{\frac{\exp\{-\lambda[(1-u_0^2)/2]t\}}{1+e^{-\lambda}} \times \left[\frac{1+\nu}{1-\nu}\exp\left(\frac{\lambda}{2}\,(1-u_0)x\right) - \exp\left(-\frac{\lambda}{2}\,(1+u_0)x\right)\right] - \frac{\nu}{1-\nu}\right\} + O(\epsilon^2)$$

$$p - p_0 = \epsilon\,\frac{\gamma Z}{\lambda}\left\{\frac{\exp\{-\lambda[(1-u_0^2)/2]t\}}{1+e^{-\lambda}} \times \left[\frac{1+\nu}{1-\nu}\exp\left(\frac{\lambda}{2}\,(1-u_0)x\right) + \exp\left(-\frac{\lambda}{2}\,(1+u_0)x\right)\right] - \frac{1}{1-\nu}\right\} + O(\epsilon^2)$$

and for region II, the solutions are

$$u - u_0 = \epsilon\,\frac{Z}{\lambda}\left\{\frac{\exp\{-\lambda[(1-u_0^2)/2]t\}}{1+e^{-\lambda}} \times \left[\frac{1+\nu}{1-\nu}\exp\left(\frac{\lambda}{2}\,(1-u_0)x\right) - e^{\lambda}\exp\left(-\frac{\lambda}{2}\,(1+u_0)x\right)\right] - \frac{\nu}{1-\nu}\right\} + O(\epsilon^2)$$

$$p - p_0 = \epsilon\,\frac{\gamma Z}{\lambda}\left\{\frac{\exp\{-\lambda[(1-u_0^2)/2]t\}}{1+e^{-\lambda}} \times \left[\frac{1+\nu}{1-\nu}\exp\left(\frac{\lambda}{2}\,(1-u_0)x\right) + e^{\lambda}\exp\left(-\frac{\lambda}{2}\,(1+u_0)x\right)\right] - \frac{1}{1-\nu}\right\} + O(\epsilon^2)$$

The shock strength is calculated from the preceding relations to be as follows:

$$\Delta u_{AB} = \epsilon \frac{Z}{\lambda} \frac{1+\nu}{1-\nu} \frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\epsilon^2)$$

$$\Delta u_{BC} = \epsilon \frac{Z}{\lambda} \frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\epsilon^2)$$

$$\Delta p_{AB} = \epsilon\gamma \frac{Z}{\lambda} \frac{1+\nu}{1-\nu} \frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\epsilon^2)$$

$$\Delta p_{BC} = \epsilon\gamma \frac{Z}{\lambda} \frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\epsilon^2)$$

Also, the shock velocities are

$$V_{AB} = 1 + u_0 + \epsilon \frac{Z}{\lambda} \left\{ \frac{3-\gamma}{4} - \frac{3-\gamma}{2} \frac{\exp[-\lambda(1+u_0)t]}{1+e^{-\lambda}} \right\} + O(\epsilon^2)$$

$$V_{BC} = u_0 - 1 + \epsilon \frac{Z}{\lambda} \frac{1+\nu}{1-\nu} \times \left\{ \frac{\gamma - 3}{4} + \frac{3-\gamma}{2} \frac{\exp\{\lambda(1-u_0)[1/(1+u_0) - t]\}}{1+e^{-\lambda}} \right\} + O(\epsilon^2)$$

and the period of oscillation is

$$T = \frac{2}{1-u_0^2} + \epsilon \frac{3-\gamma}{4} \frac{Z}{\lambda} \left[ \frac{1+\nu}{1-\nu} \left( \frac{1}{1-u_0} \right)^2 + \left( \frac{1}{1+u_0} \right)^2 \right] \left\{ \frac{2}{\lambda} \frac{1-e^{-\lambda}}{1+e^{-\lambda}} - 1 \right\} + O(\epsilon^2)$$

The solution indicates a shock discontinuity followed by an exponential decay as shown in Fig. 4. For small $\lambda$, this decay is nearly linear (sawtooth type). There is no difficulty as $\lambda$ goes to zero since $(1 - e^{-\lambda})/\lambda$ stays finite. The shock strengths stay constant with time, and strength is lost or gained only in reflection. The absolute value of the shock velocity always increases in time, and the perturbation on the natural period of oscillation can be shown to be of the order of $\epsilon\lambda/12$, which is usually negligible. As $\epsilon$ goes to zero (or $\omega$ goes to one) the shock strength goes to zero. This is an important result, since it shows a consistency between linear and nonlinear results; that is, the same stability limit is predicted by both linear and nonlinear analyses. The main effect on amplitude is produced through $\epsilon$, whereas $\lambda$ has only a secondary effect on amplitude. The importance of $\lambda$ is that it indicates a definite relationship between the forcing function of the instability and the wave form of the oscillation.

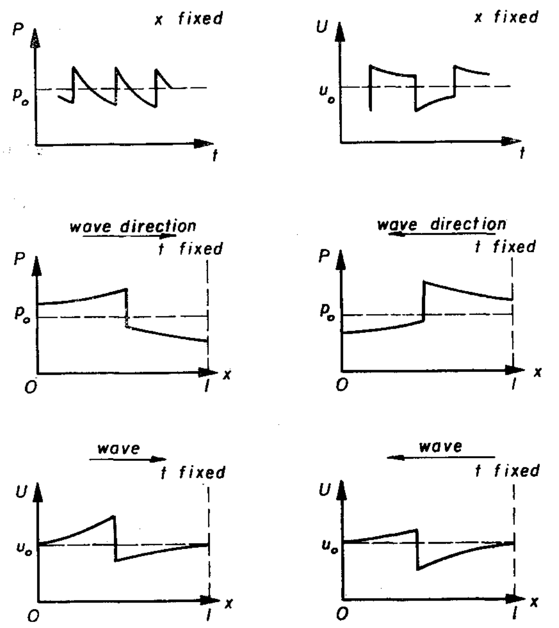


**Fig. 4 Wave shapes.**

## Investigation of Combustion Zone Dynamics

Consider a one-dimensional combustion zone near the injector of a premixed gas rocket (or the receding surface of an end-burning solid propellant rocket). It is assumed that there is always enough unreacted gas in this zone to allow the oscillation in energy release. Assuming the gas is calorically perfect, the governing equations¶ are as follows:

Equation of State

$$p = \rho T$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

Energy Equation

$$\frac{\gamma}{\gamma - 1} \rho \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left[ (\mathcal{E}_h + \lambda^*) \frac{\partial T}{\partial x} \right]$$

¶ The species equation is omitted since it is assumed that $r$ is a function of thermodynamic conditions only.

All thermodynamic variables are nondimensionalized with respect to their steady-state values at the point of completed combustion. Velocity $u$ is nondimensionalized with respect to the steady-state speed of sound at this point of completed combustion, time $t$ with respect to chamber length divided by speed of sound, space dimension $x$ with respect to chamber length, and energy release rate per unit volume $r$ with respect to steady-state pressure times speed of sound divided by chamber length. Thermal conductivity and turbulent exchange coefficient are nondimensionalized with respect to pressure times speed of sound times chamber length divided by temperature. This same nondimensionalization scheme is used in the chamber gasdynamics analysis.

The momentum equation is replaced by the assumption that the pressure gradient is zero throughout the zone. Actually the pressure gradient near the velocity node is of the order of the Mach number and is taken as negligible in small Mach number chambers. These equations may be combined to yield the following equation:

$$p \frac{\partial u}{\partial x} = -\frac{1}{\gamma} \frac{dp}{dt} + \frac{\gamma - 1}{\gamma} r + \frac{\gamma - 1}{\gamma} \frac{\partial}{\partial x} \left[ (\mathcal{E}_h + \lambda^*) \frac{\partial T}{\partial x} \right] \tag{24}$$

Equation (24) can be integrated with respect to $x$ from $x = 0$ (injection surface) to $x = l$, which is some position at which the reaction may be assumed complete. Noting that $\mathcal{E}_h = 0$ at $x = 0$ and $\partial T/\partial x = 0$ at $x = l$, the following is obtained:

$$p(u_{x=l} - u_{x=0}) = -\frac{l}{\gamma} \frac{dp}{dt} + \frac{\gamma - 1}{\gamma} \int_0^l r dx - \frac{\gamma - 1}{\gamma} \left[ \lambda^* \frac{\partial T}{\partial x} \right]_{x=0}$$

which may be rewritten as

$$p u_{x=l} = -\frac{l}{\gamma} \frac{dp}{dt} + \frac{\gamma - 1}{\gamma} \int_0^l r dx + \left[ \dot{m} T - \frac{\gamma - 1}{\gamma} \lambda^* \frac{\partial T}{\partial x} \right]_{x=0} \tag{25}$$

Equation (25) may be simplified by assuming that the heat transferred to the injector surface (or solid propellant surface) and the convected energy through that surface are negligibly small compared to the energy of combustion. (Note that this allows no change in the mean burning rate.)

Letting zero subscript denote steady-state values, the steady-state equation states

$$u_{0_{x=l}} = \frac{\gamma - 1}{\gamma} \int_0^l r_0 dx \tag{26}$$

Allowing primes to denote perturbation quantities, Eq. (26) is subtracted from (25) to yield

$$(pu)'_{x=l} = -\frac{l}{\gamma}\frac{dp'}{dt} + \frac{\gamma - 1}{\gamma}\int_0^l r' dx \tag{27}$$

$(pu)'_{x=l}$ represents the energy put into the oscillation of the chamber gases. According to Eq. (27), this is large when the reaction zone is spread out in distance (large $l$). In other words, the rocket becomes more unstable as the reaction zone is lengthened. It is stressed, therefore, that according to this model, lower chamber pressures would cause a more unstable situation.

The energy release rate is a function of temperature and pressure** and may be expanded in a double Taylor series about the steady-state values, so that

$$r = r_0\left\{1 + \left(\frac{1}{r}\frac{\partial r}{\partial p}\right)_0 (p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_0 (T-1) + \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_0 (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p \partial T}\right)_0 (p-1)(T-1) + \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)_0 (T-1)^2 + \cdots\right\} \tag{28}$$

Substituting Eq. (28) into Eq. (25) and making use of Eq. (26), the result is

$$\left(\frac{u-u_0}{u_0}\right)_{x=l} = -(p-1) - (p-1)\left(\frac{u-u_0}{u_0}\right)_{x=l} - \frac{1}{u_0}\frac{l}{\gamma}\frac{dp}{dt} + \frac{\int_0^l r_0\left[\left(\frac{1}{r}\frac{\partial r}{\partial p}\right)_0 (p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_0 (T-1)\right] dx}{\int_0^l r_0 dx} + \frac{\int_0^l r_0\left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_0 (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p\partial T}\right)_0 (p-1)(T-1) + \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)_0 (T-1)^2\right] dx}{\int_0^l r_0 dx}$$

Now, it shall be assumed that the reaction zone is so thin that the time derivative term is negligible. Also, the integrals shall be replaced by using mean values of $r$, its derivatives, and $T$ in the combustion zone. This yields

$$\left(\frac{u-u_0}{u_0}\right)_{x=l} = -(p-1) + \left[\left(\frac{1}{r}\frac{\partial r}{\partial p}\right)_0 (p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_0 (T-1)\right]_M + \left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_0 (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p\partial T}\right)_0 (p-1)(T-1) + \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)(T-1)^2\right]_M - (p-1)\left(\frac{u-u_0}{u_0}\right)_{x=l} \tag{29}$$

where the subscript $M$ denotes a mean value. Note that $r_0$ possesses a maximum somewhere in the combustion zone and that the mean values are approximately the local values at some point near the maximum point.

Now, of course, the temperature perturbation $(T - 1)$ must be related to the pressure perturbation $(p - 1)$. The relationship between these two perturbations differs from the isentropic relationship because of two effects: release of energy and diffusion within the reaction zone. Diffusion effects are negligible when the diffusion relaxation time is much longer than the period of oscillation; however, the chemical relaxation time is almost always much shorter than the period of the oscillation. Rather than attempting a solution of the nonlinear, parabolic equation, which would give the correct relationship, the isentropic relationship shall be assumed. Therefore, diffusion relaxation time is assumed very long, and variations in entropy production due to combustion are deliberately neglected. Note that this isentropic relationship will apply through second-order in wave amplitude even when a shock wave moves through the combustion zone. Under these assumptions, there will be no time lag or delay in the system.

It is convenient to relate $(p-1)$ and $(T-1)$ to $(a-1)$ through the isentropic relationship obtaining the following in series form:

$$\begin{gathered}(p-1) = \frac{2\gamma}{\gamma - 1}(a-1) + \frac{\gamma(\gamma+1)}{(\gamma-1)^2}(a-1)^2 + \cdots \\ (T-1) = 2(a-1) + (a-1)^2\end{gathered} \tag{30}$$

Substituting Eqs. (30) into Eq. (29), the following result is obtained:

$$\left(\frac{u-u_0}{u_0}\right)_{x=l} = \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_0 \times (a-1) - \left(\frac{u-u_0}{u_0}\right)_{x=l}\frac{2\gamma}{\gamma-1}(a-1) + \left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} - \frac{\gamma(\gamma+1)}{(\gamma-1)^2} + 2\left(\frac{\gamma}{\gamma-1}\right)^2 \times \frac{1}{r}\frac{\partial^2 r}{\partial p^2} + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p\partial T} + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right]_0 (a-1)^2 + \cdots$$

It is understood that mean steady-state values are used for the coefficients of $(a - 1)$, $(a - 1)^2$, etc. These are the same mean values as appeared in Eq. (29).

Now, using the method of successive approximations to substitute for $[(u - u_0)/u_0]_{x=l}$ on the right-hand side of this equation, the final relation becomes

$$\left(\frac{u-u_0}{u_0}\right)_{x=l} = \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_0 \times (a-1) - \frac{2\gamma}{\gamma-1}\left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_0 (a-1)^2 + \left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} + 2\left(\frac{\gamma}{\gamma-1}\right)^2\frac{1}{r}\frac{\partial^2 r}{\partial p^2} + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p\partial T} + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2} - \frac{\gamma(\gamma+1)}{(\gamma-1)^2}\right]_0 (a-1)^2 + \cdots \tag{31}$$

** In general, the energy release rate is a function of the time derivatives of pressure and temperature as well. However, it is assumed that the reaction kinetics are sufficiently rapid so that these effects are negligible.

Equation (31) is a nonlinear relation which states, in an approximate manner, the perturbation in the outflow velocity from the combustion zone due to a perturbation in the mean thermodynamic conditions within the combustion zone. Therefore, this is the governing relation for the feedback of energy to the instability.

By comparison with Eq. (9), it is seen that for this type of combustion process

$$\left.\begin{aligned}\omega &\equiv \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_0 \\ \delta &\equiv -\frac{2\gamma\omega}{\gamma-1} + \left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} + 2\left(\frac{\gamma}{\gamma-1}\right)^2\frac{1}{r}\frac{\partial^2 r}{\partial p^2} + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p\partial T} + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right]_0 - \frac{\gamma(\gamma+1)}{(\gamma-1)^2}\end{aligned}\right\} \quad (32)$$

## Numerical Example

A special case of the energy release rate $r$ will be examined. In this case

$$r = e^{-E/RT^*}$$

where $E/R$ is a parameter and $T^*$ is considered as dimensional. This rate function admittedly may be too simple to be realistic, but is primarily intended to show the interesting relationship between functional forms of the combustion laws and the flow oscillation in a quantitative manner.

By use of Eqs. (10, 22, and 32), $\epsilon$ and $\lambda$ may be calculated. The following are obtained:

$$\epsilon = u_0\left[(\gamma-1)\frac{E}{RT^*} - \frac{3\gamma-1}{2}\right]$$

$$\lambda = \frac{16u_0\left(\frac{\gamma-1}{2}\frac{1}{1-\nu}\right)^2\left[2\left(\frac{E}{RT^*}\right)^2 - \frac{7\gamma-3}{\gamma-1}\frac{E}{RT^*} + \frac{\gamma(3\gamma-1)}{(\gamma-1)^2}\right]}{(\gamma+1)(1+\nu)\left[\frac{1-u_0}{1+u_0}\frac{1+\nu}{1-\nu} + \frac{1+u_0}{1-u_0}\right]}$$

Using the values $\gamma = 1.2$ and $u_0 = 0.1$, the numerical results are given in Table 1. For this case, the combustor is most unstable for low-temperature operation and most stable in high-temperature operation. The amplitudes are not excessively large, and the exponential shape of the wave is not too severe despite the exponential form of the combustion law. Figure 5 indicates the pressure wave shape at both ends of the chamber over the period of oscillation for the values of $\lambda = 0.6$ and 1.2. Note that the shape is not very different from a sawtooth wave.

Although calculations have been made here only for a very simple case, there is no reason why they can't be accomplished for more realistic combustion processes.

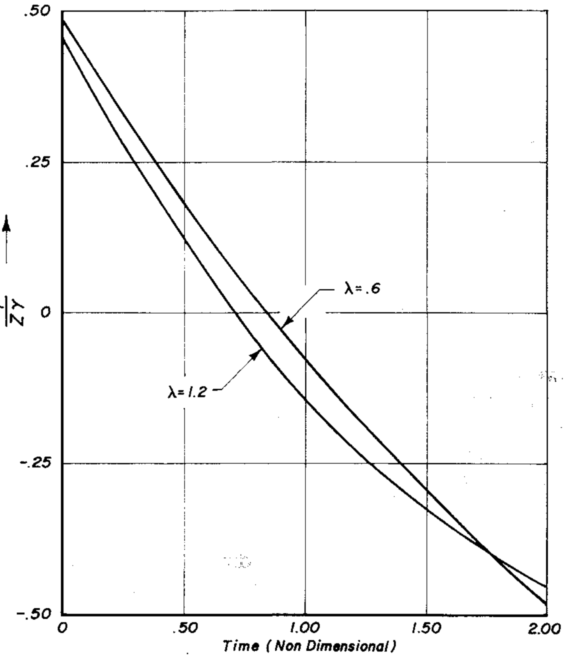


Fig. 5 Pressure wave shape at chamber end.

## Discussion and Comparison with Experiments

The best experiment for comparison with this theory seems to be the Princeton gas rocket. Details of that research are presented in Ref. 8. The gas rocket is of variable length and small diameter and burns premixed (hydrogen-air) gases with $E/RT^*$ estimated at about 10 near the stability limits. Only the longitudinal mode of instability occurs, and it is observed in the form of shock waves followed by small exponential decay in pressure. Figure 6 shows the pressure wave form observed with the Princeton gas rocket. The pressure jump across the shock is of the order of those calculated in the simplified example. At low lengths, the engine

Table 1

| $E/RT^*$ | $\epsilon$ | $\lambda$ | $Z$ | $\Delta p_{AB}$ |
|---|---|---|---|---|
| 10.0 | 0.07 | 0.028 | 3.53 | 0.152 |
| 12.5 | 0.12 | 0.189 | 3.53 | 0.259 |
| 15.0 | 0.17 | 0.439 | 3.53 | 0.362 |
| 17.5 | 0.22 | 0.778 | 3.53 | 0.453 |
| 20.0 | 0.27 | 1.207 | 3.53 | 0.522 |

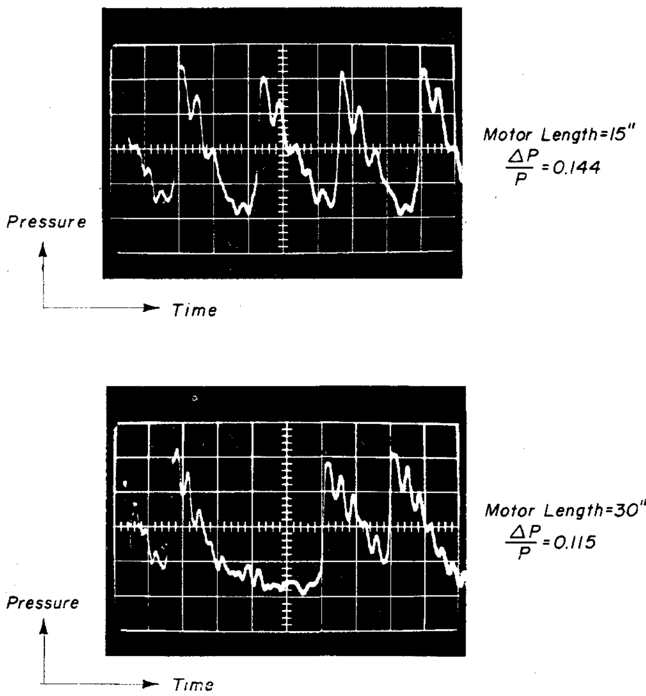


Fig. 6 Experimental results from Princeton gas rocket.

is stable, but at longer lengths, where the concentrated combustion at injector and short nozzle assumptions are reasonable, instability occurs (at the preferred mixture ratios) with no upper length limit. The preferred mixture ratios for instability occur away from and on both sides of stoichiometric such that the same temperature is measured at both stability limits. The various data collected in that program seem to be leading to the conclusion that chemical kinetics provides the driving mechanism for the instability.

The theory predicts that shock discontinuities always occur (whenever the operation is unstable) no matter how small the amplitude of the oscillation. This agrees with the experimental results of the Princeton gas rocket. On the other hand, if a phase or time delay of sufficient magnitude exist between pressure oscillation and energy release oscillation, an analysis shows that oscillations without shock discontinuities are possible.[11]

A completely accurate analysis of the combustion zone dynamics for the gas rocket would require use of the rate function $\rho^n e^{-E/RT^*}$. The main difficulty here would be in determining how the gas density in the combustion zone behaves under oscillating conditions. If the isentropic relation were used, the rocket would be unstable for values of $n = 1$ or greater regardless of the exponential effect. Stable operation over a certain range would be possible only with values of $n$ less than unity. However, energy release (and diffusion effects if diffusion time is not too long) make the isentropic condition invalid. The proper relation could only be obtained by solution of the unsteady heat equation (which has not yet been done for this situation).

If it is assumed that the gas rocket combustion process behaves in similar fashion to the model investigated earlier in this paper, and, more specifically, that the energy release rate $r$ follows an Arrhenius type law approximated sufficiently well for our purposes by the numerical example, many of the observed phenomena can be explained by this theory. Using the estimated values of $E/RT^*$ for the gas rocket, the theory predicts instability consisting of shock waves followed by small exponential decay and also predicts the criticality of the temperature at the stability limits, since the amplitude is linearly dependent upon $E/RT^*$. The theory (which neglects friction dampening) predicts no upper length limit. The theory predicts instability at all lengths (for favorable temperature); however, the assumptions of the theory are violated at lower lengths. At short lengths the combustion zone is nearer the pressure node†† of an acoustic oscillation, which tends to be stabilizing and explains the lower length-limit of instability observed in the rocket.

An important observation to be made is that the characteristic time of the forcing function of the instability need not be of the same order of magnitude as the period of oscillation. In this model of the gas rocket, chemical kinetics provides the forcing function for the instability, and the characteristic chemical relaxation time is many orders of magnitude smaller than the wave travel time. This does not mean, however, that in some actual situations, the characteristic time of the forcing function is not critically important.[9–11]

In a series of tests performed by Crocco and Harrje[12] using like-on-like injection in a liquid propellant rocket motor of variable length, results similar to those of the gas rocket were obtained if there were end-impingement of the spray fans of like propellants. That is, the instability regions occurred at low and high mixture ratios with no clear upper-length limit. If there were no end-impingement, an external pulsing technique was necessary to move the engine into unstable operation. The unstable operation occurred at all mixture ratios with no upper-length limit when the engine was pulsed. The instability occurred in the longitudinal mode, and the waveforms were shock waves followed by exponential decays. Because of large mass flow per unit area, the amplitudes were quite high, and it is questionable that a theory such as this one, which involves a series expansion in an amplitude parameter, could be applicable in a quantitative manner. However, qualitative comparison might be possible on the basis of more definitive experiments.

Longitudinal instability consisting of shock waves followed by rapid decay of pressure has been observed in radially burning solid propellant motors by Brownlee[13] and Dickinson.[14] In these motors, burning occurred over the length of the chamber. It is interesting, therefore, that even though the condition of concentrated combustion is violated, this type of instability is still observed.

One can see how, for all types of rocket engines, a relationship should exist between the wave form in the longitudinal mode of instability and the energy release rate as a function of position in the chamber. Presently, it is possible to show this relationship analytically only for concentrated combustion at the chamber end. In principle, however, this concept can be extended to other configurations. However, for the sole purpose of seeking information concerning the combustion process, an engine may be constructed which approximately satisfies the assumptions of the theory, and the wave form of the instability may be observed by use of pressure transducers. This promises to be a powerful technique for the study of combustion processes. Of course, in applying this information to other configurations, it would be assumed that the combustion process remains unchanged.

## Appendix A: Small Perturbation Analysis

The stability of the steady-state operation is examined for the configuration as already defined, that is, a one-dimensional flow in a constant-area chamber with a zero-length nozzle and a combustion zone concentrated at the chamber end. The transformation to characteristic coordinates is made with the following numbering system for the coordinates: $\alpha = \beta$ at $x = 0$ (combustion zone), $\alpha = 1 + \beta$ at $x = 1$ (nozzle), and $\alpha = \beta = 0$ at the initial point $x = t = 0$. The equations of the system are given by Eq. (2).

For the purposes of this analysis, the first two equations are of main interest. That is, it is desired to know the time-wise behavior of a small perturbation of the flow properties. Nonlinear corrections in the coordinates are not desired. Therefore, the first two equations only are investigated in order to determine the stability of the system under the influence of small perturbations.

These equations have the following solutions:

$$P(\beta) = [2/(\gamma - 1)]a + u$$

$$Q(\alpha) = [2/(\gamma - 1)]a - u$$

All flow properties are considered as a steady-state term plus a perturbation term:

$$P(\beta) = P_0(\beta) + P'(\beta)$$

$$Q(\alpha) = Q_0(\alpha) + Q'(\alpha)$$

$$u = u_0(\alpha, \beta) + u'(\alpha, \beta)$$

$$a = 1 + a'(\alpha, \beta)$$

where zero subscripts denote steady-state quantities, and primed symbols indicate perturbation values.

The perturbation equations are found to be the following:

$$P'(\beta) = [2/(\gamma - 1)]a' + u'$$

$$Q'(\alpha) = [2/(\gamma - 1)]a' - u'$$

so that

†† This means the combustion is not concentrated at the chamber end but instead distributed over a significant portion of the chamber length.

$$u' = [P'(\beta) - Q'(\alpha)]/2$$
$$a' = [(\gamma - 1)/4][P'(\beta) + Q'(\alpha)] \tag{A1}$$

The boundary condition at the combustion zone results from perturbing Eq. (9). It may be written as follows:

$$u' = \omega u_0 a' \text{ at } \alpha = \beta$$

Substitution from Eq. (A1) results in the following relationship:

$$(1 - \omega\nu)P'(\beta) = (1 + \omega\nu)Q'(\beta) \tag{A2}$$

where

$$\nu = (\gamma - 1)u_0/2$$

Equation (A2) indicates that a wave gains strength in reflection at the combustion zone ($P'/Q' > 1$). This gain is expected because of energy addition from the combustion process.

The boundary condition at the nozzle entrance is written

$$u' = u_0 a' \text{ at } \alpha = \beta$$

For this case, substitution from Eq. (A1) yields the relationship

$$(1 - \nu)P'(\beta) = (1 + \nu)Q'(1 + \beta) \tag{A3}$$

Equation (A3) indicates that a wave loses strength in reflection at the nozzle ($Q'/P' < 1$). This loss results from the convection of energy of oscillation out of the chamber through the nozzle. Note that changes in wave strength may occur only at the nozzle or combustion zone.

Now, Eqs. (A2) and (A3) may be combined to yield the following relationship:

$$\frac{P'(1+\beta)}{P'(\beta)} = \frac{Q'(1+\beta)}{Q'(\beta)} = \left(\frac{1-\nu}{1-\omega\nu}\right)\left(\frac{1+\omega\nu}{1+\nu}\right) \equiv \mu$$

Note that the natural period of oscillation equals unity (in characteristic coordinates). Therefore, the value of $\mu$ indicates the change in amplitude of the disturbance over a period of time. Specifically, whenever $\mu > 1$, exponential growth of the disturbance occurs, and whenever $\mu < 1$, exponential decay occurs. In the special case $\mu = 1$, a neutral oscillation of the small disturbance is obtained.

The following is readily seen: $\omega > 1$ implies $\mu > 1$, $\omega = 1$ implies $\mu = 1$, and $\omega < 1$ implies $\mu < 1$. Therefore, if $\omega > 1$, the motor is unstable, i.e., small disturbances grow until inhibited by nonlinear effects. The growth is forced to occur because more energy is added to the oscillation at the combustion zone than is removed at the nozzle; whenever, $\omega < 1$, the opposite is true resulting in a stable situation, i.e., more energy is removed at the nozzle than is added at the combustion zone. Of course, whenever $\omega = 1$, the energy addition and energy removal are in balance, explaining the neutral oscillation.

## Appendix B: Investigation of Solutions to Differential Equation (21)

The differential equation that governed the waveform of the oscillation was found to be of the type

$$dy/d\alpha = (Zy + \lambda y^2)/(c - y)$$

where

$$c = \tfrac{1}{2}[y(0) + y(1)]$$

This differential equation is related to the combustion instability phenomenon by means of the following definitions:

$$y(\alpha) = Q_1(\alpha)$$

$$Z \equiv \frac{16}{\gamma+1}\,\frac{1}{(1-\nu)^2\left[\dfrac{1-u_0}{1+u_0}\dfrac{1+\nu}{1-\nu} + \dfrac{1+u_0}{1-u_0}\right]}$$

$$\lambda \equiv \frac{16u_0\delta}{(\gamma+1)(1+\nu)}\,\frac{\{[(\gamma-1)/2][1/(1-\nu)]\}^2}{\dfrac{1-u_0}{1+u_0}\dfrac{1+\nu}{1-\nu} + \dfrac{1+u_0}{1-u_0}}$$

The value of $Z$ is always positive for practical situations. $\lambda$ is usually positive, but may be negative. A continuous solution for $y$ ($\alpha$) is sought in the range $0 \le \alpha \le 1$. Since expansion shocks are not allowed, negative jumps in $y$ (or $Q_1$) are ruled out. Positive jumps (which correspond to compression shocks) are allowed at the endpoints (0 and 1). Since $c$ is defined as the average of the values of $y$ at the two endpoints and the solution is continuous between the endpoints, there must exist some value of $\alpha$ (in the given range) such that $y = c$ at that value.

It is more convenient to look at the reciprocal equation

$$d\alpha/dy = (c - y)/(Zy + \lambda y^2) \tag{B1}$$

Note that if $\alpha(y)$ is a solution, $\alpha(y)$ plus any constant is also a solution. Thus, any curve of the solutions may be translated in the $\alpha$ direction on an $\alpha$ vs $y$ plot in order to satisfy one condition on the solution.

The topological structure depends very much on the values of $\lambda$, $Z$, and $c$. The case $c > 0$ and $\lambda c + Z > 0$ is considered first. Here, a maximum of $\alpha(y)$ exists at $y = c$. It is also seen that $\alpha$ becomes infinite at $y = 0$ and $y = -Z/\lambda$. There are three separate structures within this first case depending upon whether $\lambda$ is greater than, equal to, or less than zero.

If $\lambda > 0$, the following additional information is given by the differential equation

| | |
|---|---|
| $y < -(Z/\lambda)$ | $d\alpha/dy > 0$ |
| $0 > y > -(Z/\lambda)$ | $d\alpha/dy < 0$ |
| $0 < y < c$ | $d\alpha/dy > 0$ |
| $y > c$ | $d\alpha/dy < 0$ |

The solutions are continuous within each of three adjacent regions. They cannot, however, be continued from one region to the next except, perhaps, at infinity.

In the subcase $\lambda = 0$, the differential equation indicates the following:

| | |
|---|---|
| $y < 0$ | $d\alpha/dy < 0$ |
| $0 < y < c$ | $d\alpha/dy > 0$ |
| $y > c$ | $d\alpha/dy < 0$ |

Note that there are two continuous regions with a discontinuity between them. The results for the subcase $\lambda < 0$ are obtained in a similar fashion. It is seen that they are similar to the other subcases. The only possibility here is that $c < -Z/\lambda$ (since $c > 0$, $\lambda c + Z > 0$, $Z > 0$, and $\lambda < 0$).

Next, the case $c > 0$ and $\lambda c + Z < 0$ is examined. The only possibilities here are $\lambda < 0$ and $c > -Z/\lambda$. A minimum is found at $y = c$ and infinite slopes are indicated at $y = 0$ and $y = -Z/\lambda$. In addition, the differential equation indicates that

| | |
|---|---|
| $y < 0$ | $d\alpha/dy < 0$ |
| $0 < y < -(Z/\lambda)$ | $d\alpha/dy > 0$ |
| $-(Z/\lambda) < y < c$ | $d\alpha/dy < 0$ |
| $y > c$ | $d\alpha/dy > 0$ |

Three regions are obtained with discontinuities between them.

The third case has $c > 0$ and $\lambda c + Z = 0$. Here, $c = -Z/\lambda$, and so the only possibility is $\lambda < 0$. The slope is infinite at $y = 0$ and at first sight is indeterminate at $y = c$. Application of L'Hospital's rule, however, shows that $d\alpha/dy = 1/Z$ at $y = c$. In this case, no maxima or minima exist except at infinity. The slope is positive for $y > 0$ and negative for $y < 0$.

Whenever $c > 0$ and $\lambda c + Z \neq 0$, it is not possible to have a continuous solution in the range $0 \leq \alpha \leq 1$. Since $c \equiv \frac{1}{2}[y(0) + y(1)]$ and $y(1) < y(0)$ for a compressive shock, it is necessary that $y(0) > c$ and $y(1) < c$. It is not possible to have a continuous solution without going outside of the range $0 \leq \alpha \leq 1$.

If $c > 0$ and $\lambda c + Z = 0$, a continuous solution can be found between $y(1)$ and $y(0)$. The slope is positive, however, so that $y(1) > y(0)$, which would indicate an expansion shock.

The possibility $c > 0$ is ruled out, therefore, by not allowing discontinuities within the range $0 \leq \alpha \leq 1$ and by not allowing expansion shocks.

The sketches of the solutions for all the subcases within the case $c < 0$ are obtained in similar fashion. Identical arguments may be used to rule out the possibility $c < 0$.

Finally, it can be shown that, in the case $c = 0$, the restrictions on the solution are not violated. In that case, the Eq. (B1) becomes

$$d\alpha/dy = -1/(\lambda y + c) \qquad \text{(B2)}$$

There are three subcases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. If $\lambda \neq 0$, an infinite slope occurs at $y = -Z/\lambda$. Whenever $\lambda = 0$, a straight line is obtained. In each of the three subcases, a continuous solution may be found with $y(0) > c = 0 > y(1)$. It is concluded, therefore, that $c = 0$ is the only possibility.

Now, it remains to solve Eq. (B2) with the condition $y(1) = -y(0)$. This statement of a relationship between the solutions at two points is just as satisfactory in determining the arbitrary constant as is the statement of the exact value of the solution at one point. Equation (B2) may be integrated to yield the following results:

$$\alpha = -\int_{y(0)}^{y} \frac{dy}{\lambda y + Z} = -\frac{1}{\lambda}\left(\frac{\lambda y + Z}{\lambda y(0) + Z}\right)$$

This may be rewritten in a more convenient form as follows:

$$y(\alpha) = -\frac{Z}{\lambda} + \left[\frac{Z}{\lambda} + y(0)\right] e^{-\lambda\alpha} \qquad \text{(B3)}$$

Setting $\alpha = 1$, it is found that

$$y(0) = -y(1) = \frac{Z}{\lambda}\frac{1 - e^{-\lambda}}{1 + e^{-\lambda}}$$

Substituting the preceding for $y(0)$ in Eq. (B3), the final solution is

$$Q_1(\alpha) = y(\alpha) = \frac{Z}{\lambda}\left[\frac{2}{1 + e^{-\lambda}} e^{-\lambda\alpha} - 1\right]$$

Taking the limit as $\lambda \to 0$, a linear relation is found:

$$Q_1(\alpha) = Z(\tfrac{1}{2} - \alpha)$$

This same result could have been obtained by setting $\lambda = 0$ in the differential equation (B2) and then integrating.

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